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DIFFRACTION OF A SHORT ACOUSTIC WAVE BY A SMOOTH BODY WITH A DISCONTINUITY IN THE RADIUS OF CURVATURE OF ITS SURFACE*

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The propagation of a short acoustic wave in an ideal fluid when the radius of curvature of the wave front is discontinuous is considered. Such a wave arises if a short acoustic wave with a continuous radius of curvature is reflected from a smooth body whose surface has a discontinuity in the radius of curvature. The size of the body and its radius of curvature are assumed to be much greater than the wavelength.

In the immediate proximity of a body, an incident wave is reflected as a locally plane wave according to the laws of geometrical acoustics. Further from the body, geometrical convergence or divergence of rays begins to have an effect, and this determines the wave dynamics. If one of the radii of curvature of the body has a discontinuity along a line, the radius of curvature of the wave front also has a discontinuity, which lies on rays that originate from the points of the radius-of-curvature discontinuity line on the body surface. The geometrical acoustics solution produces different values of the wave amplitude on different sides of these rays, i.e., it has a strong tangential discontinuity and is thus inapplicable in the neighbourhood of rays that correspond to the curvature discontinuity of the wave front; diffraction of the reflected wave is observed in this region. We will derive a solution that describes the reflected wave everywhere, including the diffraction zone. The solution is obtained by matching asymptotic expansions, a method which has been previously applied to a number of other problems /1, 2/. The transverse profile of the wave is arbitrary and it is only required to satisfy the condition of zero perturbations on the leading characteristic.

Different wave-front geometries are possible. If the front is convex on both sies of the discontinuity, the diffraction zone goes to infinity. An interesting application of this problem is the design of a focusing reflector with a rounded edge. In this case, the intensity of the wave reflected from the concave reflector increases near the focus, while the intensity of the wave reflected from the convex edge decrease. The diffraction zone where these two geometrical acoustics solutions are matched may play an important role in flow calculations in the focal zone, because the opening angle of the focused wave decreases as we approach the focus while the diffraction zone increases. Our solution makes it

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possible to describe this process and to find the asymptotic behaviour of the converging wave allowing for the reflector edge effect, which is needed to construct the solution in the focal zone.

A wave front with a discontinuous radius of curvature can also form due to the action of a radiator of a similar shape on a fluid.

The advantage of the proposed solution is that it does not necessitate expanding the original wave in a superposition of harmonics. The reduction of the unsteady problem to a steady problem has a number of drawbacks. First, the inverse integral transform is a complex operation, and second, the final solution virtually eliminates the zone of dependent perturbations, which is characteristic of hyperbolic equations. This is an important point, especially because we repeatedly use passage to the limit in order to determine the asymptotic expansions. If the profile of the original wave is a function with a compact support, say, then it can be continued by an non-zero function. In the linear approximation, this continuation does not affect the solution that propagates along the characteristics originating from the points of the initial profile. Yet the expansion coefficients in Fourier or Laplace series depend on the complete distribution of the parameters in the original wave, including its arbitrary continuation.

1. Statement of the problem. Fluid flow is described by a system of Euler equations. Introducing the characteristic length of the incident wave $\bar{x}_0$ and the characteristic velocity $\bar{U}_0$, we change to dimensionless variables and define a small parameter ε:

$$x=\frac{\bar{x}}{\bar{x}_0},\quad y=\frac{\bar{y}}{\bar{x}_0},\quad z=\frac{\bar{z}}{\bar{x}_0},\quad \mathbf{U}=\frac{\bar{\mathbf{U}}}{\bar{U}_0},\quad t=\frac{\bar{t}\bar{U}_0}{\bar{x}_0},\quad p=\frac{\bar{p}}{\bar{\rho}_0\bar{U}_0^2},$$
$$\rho^0=\frac{\bar{\rho}}{\bar{\rho}_0},\quad \varepsilon=\left(\frac{\partial p}{\partial\rho^0}\right)_0^{-1}$$

where $\bar{x}, \bar{y}, \bar{z}, \bar{\mathbf{U}}$ are the Cartesian coordinates and the velocity in this system of coordinates, t is the time, $\bar{p}$ is the pressure, $\bar{\rho}$ is the density, and $\bar{\rho}_0$ is the density of the unperturbed medium. We deine ρ by the formula $\rho^0 = 1 + \varepsilon\rho$.

After linearization with respect to ε for low-intensity waves, the system of Euler equations takes the form

$$\partial\mathbf{U}/\partial t = -\nabla\rho,\quad \varepsilon\partial\rho/\partial t + \nabla\mathbf{U} = 0 \tag{1.1}$$

We assume that the size of the body is much greater than the characteristic length of the incident wave. In this case, both the incident and the reflected waves in the neighbourhood of the body may be regarded as locally plane waves obeying the laws of reflection of geometrical optics: the phase distribution is preserved, and only the direction of propagation of the wave changes.

In the neighbourhood of the body, the reflected wave front has the shape of a surface with a discontinuous radius of curvature. The equation of this surface is written in the form $\mathbf{r} = \mathbf{r}^*(\beta, \gamma)$ $(\mathbf{r} = (x, y, z))$, where β and γ are surface coordinates. We introduce the ray coordinates β, γ, σ, t:

$$\mathbf{r} = \mathbf{r}^*(\beta,\gamma) + \mathbf{n}^*(\beta,\gamma)\sigma,\quad t = t,\quad \mathbf{n}^* = \frac{[\partial\mathbf{r}^*/\partial\beta\times\partial\mathbf{r}^*/\partial\gamma]}{|[\partial\mathbf{r}^*/\partial\beta\times\partial\mathbf{r}^*/\partial\gamma]|}$$

We choose the coordinate lines $\beta = \text{const}$ and $\gamma = \text{const}$ to coincide with the principal directions of the initial reflected wave front. Then the principal curvatures of this surface are given by

$$R_1 = \frac{g^*_{\beta\beta}}{|\partial\mathbf{r}^*/\partial\beta||\partial\mathbf{n}^*/\partial\beta|},\quad R_2 = \frac{g^*_{\gamma\gamma}}{|\partial\mathbf{r}^*/\partial\gamma||\partial\mathbf{n}^*/\partial\gamma|}$$
$$g^*_{\beta\beta} = (\partial\mathbf{r}^*/\partial\beta)^2,\quad g^*_{\gamma\gamma} = (\partial\mathbf{r}^*/\partial\gamma)^2$$

The system of Eqs.(1.1) in ray coordinates takes the form

$$\frac{\partial U^1}{\partial t} = -\frac{\partial\rho}{\partial\beta}\frac{1}{g_{\beta\beta}},\quad \frac{\partial U^2}{\partial t} = -\frac{\partial\rho}{\partial\gamma}\frac{1}{g_{\gamma\gamma}},\quad \frac{\partial U^3}{\partial t} = -\frac{\partial\rho}{\partial\sigma} \tag{1.2}$$
$$\varepsilon\frac{\partial\rho}{\partial t} + \frac{\partial U^1}{\partial\beta} + \frac{\partial U^2}{\partial\gamma} +$$

$$\frac{\partial U^3}{\partial \sigma} + U^1(\Gamma_{11}+\Gamma_{21}) + U^2(\Gamma_{12}+\Gamma_{22}) + U^3(\Gamma_{13}+\Gamma_{23}) = 0$$

$$g_{\beta\beta} = \left(1 \pm \frac{\sigma}{R_1}\right)^2, \quad g_{\gamma\gamma} = \left(1 \pm \frac{\sigma}{R_2}\right)^2; \quad \Gamma_{k1} = \pm\left(1 \pm \frac{\sigma}{R_k}\right)^{-1} \frac{\sigma}{R_k^2} \frac{\partial R_k}{\partial \beta}$$

$$\Gamma_{k2} = \pm\left(1 \pm \frac{\sigma}{R_k}\right)^{-1} \frac{\sigma}{R_k^2} \frac{\partial R_k}{\partial \gamma}, \quad \Gamma_{k3} = \pm\left(1 \pm \frac{\sigma}{R_k}\right)^{-1} \frac{1}{R_k}$$

In formula (1.2), the lower sign corresponds to the concave front and the top sign to the convex front, and U^1, U^2 and U^3 are the contravariant velocity components in the coordinate system β, γ, σ.

The reflected wave profile in the neighbourhood of the body has the same form as the incident wave profile. Assume that it is given by the function

$$\rho = U^1\varepsilon^{-1/2} = U\varepsilon^{-1/2} = U_p(\beta, \gamma, \sigma - t\varepsilon^{-1/2}) \tag{1.3}$$

We will now determine the initial profile of the wave generated in the fluid by the action of a three-dimensional radiator that moves according to the law

$$\mathbf{r}_p = \mathbf{r}^*(\beta, \gamma) + \mathbf{n}^*\sigma_p(\beta, \gamma, t), \quad U_p = \partial\sigma_p/\partial t$$

The ordinary kinematic condition is defined on the radiator boundary.

We will construct the solution in the neighbourhood of the radiator for small times (the time is measured from the moment when the radiator starts moving), assuming that the radii of curvature of the radiator are sufficiently large: $R_k = r_k\varepsilon^{-1/2} \sim \varepsilon^{-1/2}$ $(k = 1, 2)$. The law of motion of the radiator must satisfy the condition

$$|\partial\mathbf{n}^*/\partial\beta| \sim \varepsilon^{-1/2}, \quad |\partial\mathbf{n}^*/\partial\gamma| \sim \varepsilon^{-1/2}, \quad \partial\sigma_p/\partial\beta \sim \varepsilon^{1/2},$$
$$\partial\sigma_p/\partial\gamma \sim \varepsilon^{1/2}$$

Under these assumptions, in the main approximation for $\sigma = \sigma_p(\beta, \gamma, t)$

$$U^3 = U_p$$

The small-time zone is determined by the scales

$$\sigma \sim 1, \ t = t_1\varepsilon^{1/2} \sim \varepsilon^{1/2}, \ U^3 = U \sim \varepsilon^{1/2}, \ \beta, \gamma \sim 1, \ U^1,$$
$$U^2 \sim \varepsilon, \ \rho \sim 1$$

In system (1.2), all the surface geometry terms drop out in the main approximation. Hence we obtain a wave in the form (1.3).

In the geometrical acoustics zone, the variable scales are given by

$$\beta, \gamma \sim \varepsilon^{-1/2}, \ \xi = \sigma - t\varepsilon^{-1/2} \sim 1, \ t \sim 1, \ U^1, U^2 \sim \varepsilon$$
$$U^3 = \varepsilon^{1/2}(U_1 + \varepsilon^{1/2}U_2 + \ldots), \quad \rho = \rho_1 + \varepsilon^{1/2}\rho_2 + \ldots$$

From system (1.2) we obtain a transport equation for ρ_1, using the condition for the system to be consistent in the second approximation. After that, matching the general transport solution with solution (1.3), we obtain the ordinary geometrical acoustics solution /3/

$$\rho_1 = (r_1r_2)^{1/2}(r_1 - t)^{-1/2}(r_2 - t)^{-1/2}U_p(\beta, \gamma, \xi) \tag{1.4}$$

Consider a radiator with a discontinuity in the radius of curvature for $\beta = 0$: $\lim_{\beta\to\pm 0} r_1 = r_1^{\pm}$. The radius of curvature r_2 is continuous everywhere. The system of ray coordinates acquires a singularity for $\beta = 0$. We therefore continue using the ray coordinates, but separately for the regions $\beta > 0$ and $\beta < 0$. For $\beta = 0$ the solution (1.4) has a discontinuity and a diffraction zone is formed near $\beta = 0$

2. ***Constructing the solution in the diffraction zone.*** Take the coordinates β, γ such that $g^*_{\beta\beta} = g^*_{\gamma\gamma} = 1$. Since $R_k \sim \varepsilon^{-1/2}$, the characteristic scale of variation of β on the radiator surface is $\varepsilon^{-1/2}$. Define the diffraction zone by the scales

$$\beta = \beta^*\varepsilon^{-1/4}, \ \xi \sim 1, \ t \sim 1, \ \rho = \rho_1 + \varepsilon^{1/2}\rho_2 + \ldots \tag{2.1}$$
$$U^3 = \varepsilon^{1/2}u_1 + \varepsilon u_2 + \ldots, \ U^1 = \varepsilon^{3/4}(u_1^* + \varepsilon^{1/2}u_2^* + \ldots),$$
$$U^2 \ll \varepsilon, \ \gamma \sim \varepsilon^{-1/2}$$

To fix our ideas, we will assume the wave to be concave in both principal directions, both for $\beta > 0$ and for $\beta < 0$. Substituting the expansion (2.1), we reduce system (1.2) to an equation of sound beams allowing for the geometrical convergence of the front /4/

$$\frac{\partial}{\partial\beta^*}\left[\frac{1}{(1-t/r_1)^2}\frac{\partial\rho_1}{\partial\beta^*}\right] = -2\frac{\partial^2\rho_1}{\partial t\,\partial\xi} + \frac{\partial\rho_1}{\partial\xi}\left(\frac{1}{r_1-t}+\frac{1}{r_2-t}\right) \tag{2.2}$$

Taking $r_k = r_k(\beta, \gamma)$, we obtain

$$\partial r_1/\partial\beta^* \sim \varepsilon^{1/4} \ll 1 \tag{2.3}$$

Using (2.3), we can factor out $(1 - t/r_1)^2$ from the derivative in the main approximation in (2.2), and in the diffraction zone $r_2(\beta, \gamma) = r_2(0, \gamma)$, $r_1(\beta, \gamma) = r_1^\pm(0, \gamma)$ (plus for $\beta > 0$, and minus for $\beta < 0$). Therefore, in what follows, r_2 in the diffraction zone stands for $r_2(0, \gamma)$.

Introducing new variables in the regions $\beta > 0$ and $\beta < 0$ according to the formulas $\rho_1 = \Omega^\pm[(1 - t/r_2)(1 - t/r_1^\pm)]^{-1/2}$, we obtain the equation

$$(1 - t/r_1^\pm)^{-2}\partial^2\Omega^\pm/\partial\beta^{*2} - \partial^2\Omega^\pm/\partial\xi\partial t = 0 \tag{2.4}$$

Assume that the oscillation velocity of the radiator or the tangential distribution of perturbations in the original reflected wave (1.3) is continuous over the surface. Then we have the matching conditions

$$\lim_{\beta^*\to\pm\infty}\Omega^\pm = \lim_{t\to 0}\Omega^\pm = \lim_{\beta\to 0}U_p(\beta, \gamma, \xi) = U_p(0, \gamma, \xi) \tag{2.5}$$

The condition for $\beta^* = 0$ must be added in order to close each of the problems (2.4) and (2.5). To derive the condition for $\beta^* = 0$, note that system (1.1) directly leads to a wave equation for ρ_1 in the diffraction zone. Seeing that the wave is sufficiently smooth, we use the Poisson formula to derive the condition for $\beta^* = 0$.

Define the Cartesian coordinates $x_1 = x\varepsilon^{1/2}$, $y_1 = y\varepsilon^{1/2}$, $z_1 = z\varepsilon^{1/2}$, directing the x-axis along β, the z-axis along γ at the point $(\beta = 0, \gamma, \sigma = 0)$, and the y-axis in the direction of wave propagation. Then the wave in the neighbourhood of the radiator takes the form

$$\rho_1^\pm|_{t=0} = U_p(0, \gamma, \xi^\pm), \quad \xi^\pm = y - {}^1/_2 x_1^2/r_1^\pm - {}^1/_2 z_1^2/r_2$$

In order to find the solution at the point $(0, y_1, 0, t)$, we have to integrate in Poisson's formula over the sphere $(y_1 - y_2)^2 + x_2^2 + z_2^2 = t^2$. Introducing the polar coordinates $x_2 = \mu\cos\varphi$, $z_2 = \mu\sin\varphi$, we obtain

$$\rho_1(0, y_1, 0, t) = \frac{A^+ + A^-}{4\pi t}, \quad A^\pm = \frac{\partial}{\partial t}\int\limits_{\varphi_1^\pm}^{\varphi_2^\pm} d\varphi \int\limits_0^{\mu^\pm}\frac{U_p(0, \gamma, \xi^\pm)}{t}\mu\,d\mu + \frac{1}{t}\int\limits_{\varphi_1^\pm}^{\varphi_2^\pm} d\varphi\int\limits_0^{\mu^\pm}\frac{\partial U_p(0, \gamma, \xi^\pm)}{\partial\xi}\mu\,d\mu \tag{2.6}$$

$$\varphi_1^+ = -\pi/2, \ \varphi_2^+ = \varphi_1^- = \pi/2, \ \varphi_2^- = 3\pi/2, \ \mu^{\pm 2} = (t - y_1)/\Delta^\pm$$
$$\Delta^\pm = {}^1/_2[\cos^2\varphi\,(1/t - 1/r_1^\pm) + \sin^2\varphi\,(1/t - 1/r_2)]$$

In deriving formula (2.6), we have used the fact that a short wave is locally plane, and we accordingly take the derivative with respect to ξ in the second integral.

Because of the short-wave properties of the pulse $(\mu/t \ll \mu/r_1^\pm \ll 0)$, we obtain the first terms of the phase expansion in the form

$$\xi^\pm(\mu, \varphi) = y_1 - t + \mu^2\Delta^\pm \tag{2.7}$$

From (2.6) we obtain, using (2.7),

$$A^\pm = \int\limits_{\varphi_1^\pm}^{\varphi_2^\pm} d\varphi\int\limits_0^{\mu^\pm} U_p\mu\,d\mu + U_p(0, \gamma, \xi^\pm)\int\limits_{\varphi_1^\pm}^{\varphi_2^\pm}\frac{d\varphi}{\Delta^\pm} \tag{2.8}$$
$$\xi^+ = \xi^- = \xi|_{\mu=0} = y_1 - t$$

Dropping the first integral in the main approximation in (2.8), we obtain the final condition for $\beta^* = 0$:

$$\rho_1(0, \gamma, \sigma, t) = {}^1/_2 U_p(0, \gamma, \xi)(1 - t/r_2)^{-1/2}[(1 - t/r_1^\pm)^{-1/2} + (1 - t/r_1^-)^{-1/2}] \tag{2.9}$$

Formula (2.9) is the expression of a simple geometrical fact: on rays corresponding to the discontinuity in the geometrical front, the wave amplitude is half the sum of the amplitudes obtained from the limiting values of the geometrical acoustics solutions on both sides of the discontinuity.

Given the additional condition (2.9), we can reduce the flow calculation in the diffraction zone for $\beta^* > 0$ to the following problem:

$$\partial^2\Omega_1^0/\partial\beta^{*2} + \partial^2\Omega_1^0/\partial\xi^0\partial T = 0 \quad (\xi^0 = -\xi > 0) \tag{2.10}$$
$$\Omega_1^0|_{\beta^*\to+\infty} = 0,\ \Omega_1^0|_{T=0} = 0,\ \Omega_1^0|_{\beta^*=0} = f(\xi^0, T)$$
$$\Omega_1^0 = \Omega^+ - U_p(0, \gamma, \xi),\ T = t(1 - t/r_1^+)^{-1}$$
$$f(\xi^0, T) = {}^1/_2 U_p(0, \gamma, -\xi^0)\{[(1 - t/r_1^+)(1 - t/r_1^-)^{-1}]^{1/2} - 1\}$$

Taking the Fourier sine-transform by β^*, we obtain a Goursat problem for the transform with values on the characteristics,

$$\partial^2\Omega_1^s/\partial\xi^0\partial T + \lambda^2\Omega_1^s + \lambda f(\xi^0, T) = 0 \tag{2.11}$$
$$\Omega_1^s|_{T=0} = 0,\ \Omega_1^s|_{\xi^0=0} = 0$$

The Riemann function of Eq.(2.11) is

$$v(\xi_1^0, T_1; \xi^0, T) = J_0[2\lambda\sqrt{(T - T_1)(-\xi_1^0 + \xi^0)}]$$

where J_0 is the zeroth-order Bessel function. Integrating (2.11) over the rectangle with the vertices $(0, 0)$, $(\xi^0, 0)$, (T, ξ^0), $(0, T)$, we obtain

$$\Omega_1^s(\xi^0, T) = \lambda\iint v(\xi_1^0, T_1; \xi^0, T) f(\xi_1^0, T_1)d\xi_1 dT_1$$

Making the change of variables $\lambda(T - T_1) = T_2$, $\lambda(-\xi_1^0 + \xi^0) = \xi_2$ and taking the inverse Fourier transform, we obtain the solution in the diffraction zone for $\beta^* > 0$,

$$\Omega_1^0 = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\beta^*}{\lambda}d\lambda\int_0^{-\lambda\xi} U_p\left(0, \gamma, \xi + \frac{\xi_2}{\lambda}\right)d\xi_2\int_0^{\lambda T} J_0(2\sqrt{\xi_2 T_2})\times \left\{\left[1 - \left(T - \frac{T_2}{\lambda}\right)\left(\frac{1}{r_1^+} - \frac{1}{r_1^-}\right)\right]^{-1/2} - 1\right\}dT_2 \tag{2.12}$$

To find the solution for $\beta^* < 0$, it suffices to note that substituting $\beta_1^* = -\beta^*$ we obtain for Ω_1^0 precisely the same problem (2.10) in the variables β_1^*, ξ^0, T with the sole difference that r_1^+ and r_1^- must be interchanged everywhere. The solution for $\beta^* < 0$ therefore differs from (2.12) by its sign and by interchanging r_1^+ and r_1^-.

Solution (2.12) and the corresponding solution for $\beta^* < 0$ have a removable discontinuity. Indeed, $\Omega_1^0|_{\beta^*=0} = 0$. However, the last condition in (2.10) holds if we use the limit as $\beta^* \to +0$. This is directly verified.

The solution of the problem in the diffraction zone is continuous for $\beta^* = 0$, but its tangential derivative with respect to β^* is not continuous for $\beta^* = 0$. Is the weak discontinuity a consequence of the discontinuous initial condition or a deficiency of our asymptotic solution procedure? Another question is the applicability of Poisson's formula to derive the condition for $\beta^* = 0$.

The initial values in Poisson's formula have the form

$$U|_{t=0} = U_p\left(0, \gamma, y_1 - \frac{x_1^2}{2r_1^\pm} - \frac{z_1^2}{2r_2}\right),\quad U_t'|_{t=0} = U_p'\left(0, \gamma, y_1 - \frac{x_1^2}{2r_1^\pm} - \frac{z_1^2}{2r_2}\right) \tag{2.13}$$

Hence it follows that for $\beta^* = 0$ (or $x_1 = 0$) the functions (2.13) and their first space derivatives are continuous, while the derivative

$$\left.\frac{\partial^2 U|_{t=0}}{\partial x_1^2}\right|_{x_1=0} = -{}^1/_2 U_p'\left(0, \gamma, y_1 - \frac{z_1^2}{2r_1}\right)\frac{1}{r_1^\pm}$$

has a discontinuity. At the same time, Poisson's assumes the continuity of the first three derivatives of these functions. The values (2.13) used to derive the condition (2.9) are therefore insufficiently smooth.

The initial values (2.13) can be smoothed by altering them in an arbitrarily small neighbourhood of $x_1 = 0$ so that the smoothed values are thrice differentiable and the first derivatives are arbitrarily close to continuous first derivatives of (2.13). For such smooth initial conditions, Poisson's formula produces the classical solution of the wave equation.

This argument leads to the conclusion that condition (2.9) on rays corresponding to the wave front discontinuity may be obtained by passage to the limit in smoothed initial conditions uniformly close to (2.13).

Condition (2.9) thus indeed corresponds to a discontinuous wave front. Moreover, the weak discontinuity in the solution lies on the bicharacteristic of system (1.1).

*3. **Mixed expansion.*** The mixed expansion describing both the geometrical acoustics zone and the diffraction zone has the form

$$\rho_1 = \left[\left(1-\frac{t}{r_1^\pm}\right)\left(1-\frac{t}{r_2}\right)\right]^{-1/2}\left\{U_p(\beta,\gamma,\xi) \pm \frac{1}{\pi}\int_0^\infty \frac{\sin(\lambda\beta\varepsilon^{1/4})}{\lambda}\,d\lambda\,\times\right. \tag{3.1}$$

$$\left.\int_0^{-\lambda\xi} U_p\left(0,\gamma,\xi+\frac{\xi_2}{\lambda}\right)d\xi_2\int_0^{\lambda T^\pm} J_0\left(2\sqrt{\xi_2 T_2}\right)A^\pm\,dT_2\right\}$$

$$A^\pm = [1 + (T^\pm - T_2/\lambda)(1/r_1^+ - 1/r_1^-)]^{-1/2} - 1$$

$$T^\pm = t r_1^\pm/(r_1^\pm - t)$$

Thus plus sign corresponds to $\beta > 0$ and the minus sign to $\beta < 0$.

The solution can be generalized to the case of arbitrary combinations of convexity and concavity in any coordinate; for example, the wave may be concave for $\beta < 0$, convex for $\beta > 0$, and concave in γ. The principle of the solution in this case may be formulated in geometrical terms: problem (2.10) remains as before, the wave amplitude for $\beta^* = 0$ equals half the sum of the limiting values of the geometrical acoustics solutions for any body geometry.

*4. **Investigation of the solution near the focusing zone.*** Consider the case when the wave is concave both for $\beta > 0$ and for $\beta < 0$, but $r_1^- > r_1^+$. Let $r_2 > r_1^+$, i.e., focusing is first by β for $\beta > 0$.

For $t \to r_1^+$ we have $T^+ \to \infty$, and after some substitutions in (3.1) we obtain the asymptotic expansion of ρ_1 near the focusing zone in the form

$$\rho_1 = [\Delta t\,(1/r_1^+ - 1/r_2)]^{-1/2}\,\{U_p\,(0,\gamma,\xi) - [(\Delta t/r_1^+)^{1/2}(1 - r_1^+/r_1^-)^{-1/2} - 1]G\,(R,\xi)\} \tag{4.1}$$

$$G(R,\xi) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega\int_0^D U_p\left(0,\gamma,\xi+\frac{\xi_1^2R^2}{4\omega^2}\right)\frac{d\xi_1}{\xi_1}\int_0^{\xi_1} J_0(\nu)\,\nu\,d\nu$$

$$R = \beta^* T^{+^{-1/2}},\ \Delta t = r_1^+ - t,\ D = 2\omega\sqrt{-\xi}/R$$

Solution (4.1) holds for $\beta^* \ll \varepsilon^{-1/4}$. In the main approximation, $R = \beta^*\Delta t^{1/2}/r_1^+$. Solution (4.1) therefore describes a transverse wave propagating along the wave front. Since G depends on time and the longitudinal coordinate β^* only in the combination $\beta^*\Delta t^{1/2}$, the diffraction zone for $\Delta t \to 0$ increases in proportion to $\Delta t^{-1/2}$. There is an analogy with the spread of the zone of instantaneous heat release at a point. The only difference is that the amplitude in the heat problem decreases and in the focusing problem it increases; time in the heat problem goes to infinity and here $\Delta t \to 0$. However, in both cases, the waves preserve their similarity over time, and the similarity parameter follows the same parabolic dependence $\beta^*\Delta t^{1/2} = \text{const}$. In other words, the solution (4.1) is selfsimilar: for wave focusing near the focus (but not in the focal zone itself), the transverse wave profile can be constructed by a simple change of the time-dependent coefficients, the function G is a selfsimilar.

*5. **The effect of a narrow zone of high gradients in the wave profile on diffraction.*** The wave profile $U_p\,(\beta,\gamma,\xi)$ may have a zone of high gradients in the neighbourhood of some ξ. For instance, when a shock wave propagates in a low-viscosity medium, a zone of high gradients describing the shock wave structure develops in the neighbourhood of $\xi = 0$. In order to allow for this boundary layer, we naturally need to investigate equations with viscosity. It is interesting to establish the effect of this zone in the ideal fluid framework.

The solution (3.1) describes a wave whose width is of the order of 1. To fix our ideas we will assume that the thin layer is in the neighbourhood of $\xi = 0$ and its width is of the order of $\varepsilon_1 \ll 1$. The smooth wave profile has the form

$$U_p(\beta,\gamma,\xi) = U_p^1(\beta,\gamma,\xi) + U_p^2(\beta,\gamma,\xi/\varepsilon_1)$$

where $U_p^1(\beta,\gamma,0) = -U_p^2(\beta,\gamma,0) \neq 0$, $U_p^2 = 0$ for $\xi/\varepsilon_1 < \xi_f < 0$. The function U_p^1 has a discontinuity at $\xi = 0$.

As we move away from $\xi = 0$ to a distance much greater than ε_1, we can use in the main approximation the limiting function in formulas (3.1).

To this end, we have to show that this assertion holds for the integral term in braces in (3.1). Let us estimate the integral

$$I_1 = \int_0^\infty \frac{\sin(\lambda\beta\varepsilon^{1/4})}{\lambda}\, d\lambda \int_0^{-\lambda\xi} U_p^2\left(0,\gamma,\frac{\xi+\xi_2/\lambda}{\varepsilon_1}\right) d\xi_2 \int_0^{\lambda T\pm} J_0\left(2\sqrt{\xi_2 T_2}\right) A^\pm\, dT_2 \tag{5.1}$$

After the change of variables $(\xi+\xi_2/\lambda)/\varepsilon_1 = \xi_1$, $\lambda T_2 = T_1$, we can show that for $\xi > \xi_f\varepsilon_1$ the integration over ξ_1 in the integral obtained from (5.1) should be carried out from ξ_f to zero (because U_p^2 has a compact support). Therefore, for $\xi \gg \varepsilon_1\xi_1$ we can pass to the limit as $\varepsilon_1\xi_1 \to 0$ in the integral over T_1. As a result we obtain

$$I_1 \sim \varepsilon_1 \int_{\xi_f}^0 U_p^2(0,\gamma,\xi_1)\, d\xi_1 \int_0^\infty \frac{\sin(\lambda\beta\varepsilon^{1/4})}{\lambda}\, d\lambda \int_0^{\lambda^2 T\pm} J_0\left[2\sqrt{(\varepsilon_1\xi_1 - \xi)T_1}\right] A^\pm\, dT_1$$

In view of the convergence of the integral with respect to λ, we obtain $I_1 \sim \varepsilon_1$, i.e., for $\xi \gg \xi_f\varepsilon_1$ a limiting discontinuous function may be used in the main approximation in the integral and certainly in the term outside the integral in (3.1).

6. *Focusing of a wave with a step profile.* Consider the focusing of a wave with a step profile $U_p = 1$ for $\xi < 0$ and $U_p = 0$ for $\xi > 0$. Assume that the discontinuity is smoothed in some thin layer near $\xi = 0$. As we have shown in Sect.5, the limiting discontinuous function may be used in the solution at distances much greater than the width of the smoothing zone.

Using (4.1) and taking the multiple integrals, as at the end of Sect.2, we make the change of variables $-2\lambda\sqrt{-\xi T}/\beta^* = \lambda_1$ and use the table of Fourier transforms of Bessel functions. This gives

$$\rho_1 = [\Delta t\,(1/r_1^+ - 1/r_2)]^{-1/2}\Lambda \tag{6.1}$$

$$\Lambda = 1 + [(r_1^-\Delta t)^{1/2} r_1^{+-1/2}(r_1^- - r_1^+)^{-1/2} - 1] \times (1/2 - 1/\pi^{-1}\arcsin\psi),$$

$$0 < \psi < 1,\ \beta > 0$$

$$\Lambda = 1,\ \psi > 1,\ \beta > 0$$

$$\psi = (-\Delta t/\xi)^{1/2}\beta^*/(2r_1^+)$$

The solution (6.1) clearly illustrates the following property of waves with a narrow zone of high gradients: the closer the phase ξ approaches this zone, the smaller is the diffraction zone. In the limit of an infinitely thin layer of high gradients, the solution in the neighbourhood of the boundary layers obeys the law of geometrical acoustics, which in this case produces a strong tangential discontinuity.

Consider a cylindrical converging wave with $R_2 = \infty$, $R_{11} = 0.5$ m, which is matched to the wave with $R_{12} = 0.8$ m and wave width $x_0 = 10^{-2}$ m. Define the angle $\theta = \beta^*/(R_{11}\varepsilon^{1/4})$ as shown in Fig.1. Then $\theta = 0$ is the ray where the two wave parts with different radii are matched.

Fig.1 shows the boundary of the diffraction zone $\theta_l = 2\sqrt{-\xi}\varepsilon^{1/4}/\sqrt{\Delta t}$ in polar coordinates for the case $\varepsilon = 10^{-4}$. Curves *1, 2, 3* correspond to phase values $\xi = -0.3$; -0.5; -0.8. As $|\xi|$ decreases to zero, the diffraction zones (under the curves *1, 2, 3*) also decrease and go to $\theta = 0$, although non-uniformly: for any ξ near the focus the diffraction angle θ_l may become a maximum (depending on the aperture angle of the initial wave). The curves in Fig.1 are shown up to $\theta_l = 80°$.

The existence of the finite boundary θ_l where the solution reduces to the geometrical acoustics solution is a consequence of the passage to the limit that produces (6.1). The non-uniform behaviour of the diffraction zone boundary as a function of the phase is a consequence of the high-gradient zone. We naturally expect that for slowly varying waves with gradients of the same order of magnitude, the diffraction-zone boundary will depend uniformly on ξ.

Denote by ΔR the distance from the wave to the focus. Fig.2 shows the wave profiles for $\Delta t = 0.15$ ($\Delta R = 1/3 R_{11}$, curve *1*), for $\Delta t = 0.08$ ($\Delta R = 0.16R_{11}$, curve *2*), and for $\Delta t = 0.05$ ($\Delta R = 0.1R_{11}$, curve *3*). In all cases, the phase is $\xi = -0.5$. As the wave approaches the focus, it becomes less steep. The diffraction wave reaches its limiting value (corresponding to geometrical acoustics) with a break in the graph. This drawback is the result of the passage to the limit that produces (6.1).

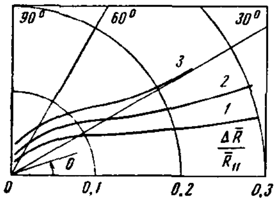

Fig.1

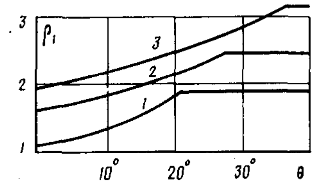

Fig.2

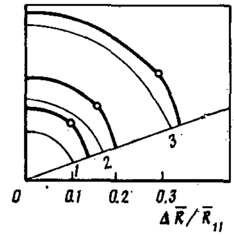

Fig.3

Fig.3 shows half of the same (symmetrical) cylindrical wave as in the previous case, in polar coordinates. The pressure ρ_1 is measured from the corresponding circles *1, 2, 3* along the normal to these circles. In dimensionless units, the intensity of the non-diffracted part of the wave in position *1* in Fig.3 equals 3. The circles *1, 2, 3* are the leading fronts of the wave at various distances from the focus. The phase is $\xi = -0.5$. We see that the focusing wave is replaced by a diffraction wave due to the wave propagating along the wave front. The size of the diffraction zone is independent of the wave geometry, i.e., of the radii $\bar{R}_{11}$ and $\bar{R}_{12}$, and is determined only by the phase and the time.

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